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The factorisation method and supersymmetry†

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Abstract. Applying the factorisation method, we generalise the harmonic oscillator and the Coulomb potentials, both in arbitrary dimensions. We also show that this method allows the determination of the superpotentials and the supersymmetric partners associated with each of those systems.

1. Introduction

The factorisation method, although well known, has recently received renewed attention. In particular we mention the papers by Mielnik (1984), Fernandez (1984) and Kimel (1982).

The fact that exactly soluble problems of quantum mechanics can be solved in terms of creation and annihilation operators has been explored. Mielnik (1984) has applied the factorisation method in the one-dimensional harmonic oscillator problem and Fernandez (1984) treated the three-dimensional Coulomb potential. In their approach, new potentials were produced. In other words, if we start with the harmonic oscillator, the procedure allows us to construct a new class of potentials which still have the oscillator spectrum. This is achieved by means of the generalised version of the operators that define the algebraic method of factorisation.

On the other hand, the realm of supersymmetric quantum mechanics has also been explored in dealing with the harmonic oscillator and Coulomb potentials. The reader is referred to the papers by Cooper and Freedman (1983, 1985), Akhoury and Comtet (1984), Kostelecky and Nieto (1984), Lancaster (1984), Ravndal (1984) and Haymaker and Rau (1986). The supersymmetric partners for those systems were obtained in arbitrary dimensions by Kostelecky *et al* (1985). Furthermore the connection between harmonic oscillators and the hydrogen atom (Bergmann and Frishman 1965, Rockmore 1975, Kibler and Negadi 1983, Cornish 1984, Kibler *et al* 1986) in arbitrary dimensions was also extended to its supersymmetric partners (Kostelecky *et al* 1985). This series of maps between the various systems involved is achieved by means of the radial solutions.

In this paper, we consider the harmonic oscillator, the hydrogen atom and their supersymmetric partners, in arbitrary dimensions, and construct the solutions by means of the factorisation method. Furthermore, we show that the generalised version of the creation and annihilation operators provided by the factorisation method leads to the possible superpotentials that define the supersymmetric charges.

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In § 2, we describe the harmonic oscillator in D dimensions by using the factorisation method. Also, through a generalised version of this method, we determine the class of potentials that exhibits the same energy spectrum as the oscillator.

In § 3, we apply the procedure to the Coulomb potential in d dimensions. We also work out the solutions of its corresponding generalised version.

In § 4, we include a descriptive discussion of the supersymmetric quantum mechanics and identify our previous Hamiltonians as components of the supersymmetric one. We also show that superpotentials can be obtained from the corresponding generalised version of the operators that naturally appear in the development of the generalised factorisation method.

2. The harmonic oscillator in arbitrary dimensions and generalised factorisation

In this section, we apply the factorisation method for the harmonic oscillator in D dimensions. The radial eigenvalue equation for the D-dimensional simple harmonic oscillator of mass m and angular frequency ω is given by

$$H_L U_{N,L}(R) = E_N U_{N,L}(R) \tag{2.1}$$

where (Kostelecky *et al* 1985)

$$H_L = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dR^2} + \frac{[L+\frac{1}{2}(D-1)][L+\frac{1}{2}(D-1)-1]}{R^2} + \frac{m^2\omega^2}{\hbar^2}R^2\right). \tag{2.2}$$

The explicit expression for $U_{N,L}(R)$ is not relevant for the purposes of this paper. Its normalisation is taken as

$$(U_{N,L}, U_{N',L'}) = \int dR\, R^{D-1} U^+_{N,L} U_{N',L'}. \tag{2.3}$$

The energy eigenvalues E_N are given by

$$E_N = \tfrac{1}{2}\hbar\omega(2N+D) \tag{2.4}$$

with $N = 2n + L$, $n = 0, 1, \ldots$, and $D \geqslant 2$.

As is well known, the factorisation method consists of introducing creation- and annihilation-type operators to obtain the radial Hamiltonian in a factorised form. For the D-dimensional harmonic oscillator, we have

$$H_L = a^+_L a_L + \tfrac{1}{2}\hbar\omega(2L+D) \tag{2.5}$$

where the operators a_L and a^+_L are given by

$$a_L = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dR} + \frac{m\omega}{\hbar}R - \frac{L+\frac{1}{2}(D-1)}{R}\right) \tag{2.6a}$$

and

$$a^+_L = \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dR} + \frac{m\omega}{\hbar}R - \frac{L+\frac{1}{2}(D-1)}{R}\right). \tag{2.6b}$$

By using these operators, we can also realise that

$$H_{L+1} = a_L a^+_L + \tfrac{1}{2}\hbar\omega(2L+D-2). \tag{2.7}$$

The operators a_L and a_L^+ are interpreted as creation and annihilation ones. In order to clarify this point, we rewrite the eigenvalue equation (2.1) as

$$H_L U_{2n+L,L}(R) = E_{2n+L,L} U_{2n+L,L}(R) \tag{2.8}$$

and

$$H_{L+1} U_{2n+L+1,L+1}(R) = E_{2n+L+1,L+1} U_{2n+L+1,L+1}(R). \tag{2.9}$$

Now it is straightforward to show that we have the relations

$$a_L a_L^+ a_L U_{2n+L,L}(R) = [E_{2n+L,L} - \tfrac{1}{2}\hbar\omega(2L+D)] a_L U_{2n+L,L}(R) \tag{2.10}$$

and

$$a_L a_L^+ U_{2n'+L+1,L+1}(R) = [E_{2n'+L+1,L+1} - \tfrac{1}{2}\hbar\omega(2L+D-2)] U_{2n'+L+1,L+1}(R). \tag{2.11}$$

By comparing these two last relations, we get a construction method for obtaining the eigenvectors and eigenvalues of H_L. From (2.10) and (2.11), we have that

$$a_L U_{2n+L,L} = U_{2n'+L+1,L+1} \tag{2.12}$$

is an eigenfunction of the operator $a_L a_L^+$, and that

$$E_{2n+L,L} = E_{2n'+L+1,L+1} + \hbar\omega \tag{2.13}$$

from which we get $n' = n-1$. Thus we conclude that

$$a_L U_{2n+L,L} = U_{2n+L-1,L+1}. \tag{2.14}$$

This relation shows us that a_L operating on $U_{2n+L,L}$ increases L by one unit while n decreases by one. Equation (2.14) is an example of a ladder relation. Similarly, we can also obtain a ladder relation involving the decreasing operator a_L^+, namely

$$a_L^+ U_{2n+L-1,L+1} = U_{2n+L,L}. \tag{2.15}$$

In the following, we will explore some possibilities of generalisation of the operators a_L and a_L^+ proposed by Mielnik (1984). In particular, we will construct a whole class of potentials in D dimensions which have the D-dimensional harmonic oscillator spectrum.

Let us define the new operators

$$A_L = \frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}R} + f_L(R)\right) \tag{2.16a}$$

and

$$A_L^+ = \frac{\hbar}{\sqrt{2m}}\left(-\frac{\mathrm{d}}{\mathrm{d}R} + f_L(R)\right) \tag{2.16b}$$

with $f_L(R)$ to be determined, and demand that the relation (2.7) also holds for these operators, namely

$$H_{L+1} = A_L A_L^+ + \tfrac{1}{2}\hbar\omega(2L+D-2) \tag{2.17a}$$

$$= \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}R^2} + f_L^2(R) + f'_L(R)\right) + \tfrac{1}{2}\hbar\omega(2L+D-2). \tag{2.17b}$$

From this condition, we obtain the following Ricatti differential equation for $f_L(R)$:

$$f'_L + f_L^2 = \frac{[L+\frac{1}{2}(D-1)][L+\frac{1}{2}(D-1)+1]}{R^2} + \frac{m^2\omega^2}{\hbar^2}R^2 - \frac{m\omega}{\hbar}(2L+D-2). \tag{2.18}$$

In the present case, from (2.6), we know a particular solution for this equation, given by

$$\tilde{f}_L(R)=\frac{m\omega}{\hbar}R-\frac{L+\frac{1}{2}(D-1)}{R}. \tag{2.19}$$

So the general solution can be written as

$$f_L(R)=\tilde{f}_L(R)+\phi_L(R). \tag{2.20}$$

Standard analytic manipulations of the resulting differential equation for $\phi_L(R)$ will produce the solution

$$\phi_L(R)=R^{2L+D-1}\exp\left(-\frac{m\omega}{\hbar}R^2\right)\left[\gamma+\int_0^R \mathrm{d}R\, R^{2L+D-1}\exp\left(-\frac{m\omega}{\hbar}R^2\right)\right]^{-1}. \tag{2.21}$$

In order to avoid problems with possible singularities in (2.21), we impose the following conditions on γ.

(i) If $(2L+D-1)$ is even then

$$\gamma>0 \qquad \text{or} \qquad \gamma<-\frac{(2L+D-2)!!}{2}\left(\frac{\hbar}{2m\omega}\right)^{(2L+D-1)/2}\left(\frac{\hbar\pi}{m\omega}\right)^{1/2}. \tag{2.22a}$$

(ii) If $(2L+D-1)$ is odd then

$$\gamma>0 \qquad \text{or} \qquad \gamma<-\frac{1}{2}\left(\frac{2L+D-2}{2}\right)!\left(\frac{\hbar}{m\omega}\right)^{(2L+D)/2}. \tag{2.22b}$$

Let us now turn to the generalised version of H_L. Let us first note that the commutator of A_L and A_L^+ is not a number, but is given by

$$[A_L, A_L^+]=\frac{\hbar}{m}\left(\frac{m\omega}{\hbar}+\frac{L+\frac{1}{2}(D-1)}{R^2}+\phi'_L(R)\right). \tag{2.23}$$

As we will see, based on this fact we will be able to define another Hamiltonian different from H_L. In this new description, we would like to establish a relation in terms of A_L and A_L^+, similar to (2.5). With this objective in mind and using (2.17) and (2.23), we write the Hamiltonian H_{L+1} in terms of A_L and A_L^+ in the following way:

$$H_{L+1}=A_L^+A_L+\frac{\hbar^2}{m}\left(\frac{m\omega}{\hbar}+\frac{L+\frac{1}{2}(D-1)}{R^2}+\phi'_L(R)\right)+\tfrac{1}{2}\hbar\omega(2L+D-2). \tag{2.24}$$

On the other hand, from (2.2) we can see that

$$H_L=H_{L+1}-\frac{\hbar^2}{m}\frac{L+\frac{1}{2}(D-1)}{R^2}. \tag{2.25}$$

Now, by comparing (2.25) with (2.24), we obtain

$$H_L-\frac{\hbar^2}{m}\phi'_L(R)=A_L^+A_L+\tfrac{1}{2}\hbar\omega(2L+D). \tag{2.26}$$

Hence, the inverted product of the operators A_L and A_L^+ produces a new Hamiltonian that differs from the previous H_L by the function $\phi'_L(R)$. Thus we are prompted to define the Hamiltonian $\mathscr{H}_L$ as

$$\mathscr{H}_L=H_L-(\hbar^2/m)\phi'_L(R) \tag{2.27}$$

which turns out to be the corresponding generalised version of H_L, namely

$$\mathscr{H}_L = A_L^+ A_L + \tfrac{1}{2}\hbar\omega(2L+D). \tag{2.28}$$

However, due to the non-commutativity of A_L and A_L^+ we obtained a new class of potentials $v(R)$ given by

$$v(R) = \tfrac{1}{2}m\omega^2 R^2 - (\hbar^2/m)\phi_L'(R). \tag{2.29}$$

The eigenvectors and eigenvalues associated with the new Hamiltonian $\mathscr{H}_L$ can be obtained by noticing that

$$\mathscr{H}_L A_L^+ = A_L^+(H_{L+1} + \hbar\omega). \tag{2.30}$$

So, if $\{U_{N,L+1}\}$ are the eigenvectors of H_{L+1}, with eigenvalues E_N then $\{A_L^+ U_{N,L+1}\}$ are the eigenvectors of $\mathscr{H}_L$ with eigenvalues $(E_N + \hbar\omega)$.

The eigenvectors of $\mathscr{H}_L$ are orthogonal for

$$\begin{aligned}(A_L^+ U_{N,L+1}, A_L^+ U_{N',L+1})\\ &= (A_L A_L^+ U_{N,L+1}, U_{N',L+1})\\ &= [-\tfrac{1}{2}\hbar\omega(2L+D-2) + E_N]\,\delta_{N,N'}.\end{aligned} \tag{2.31}$$

However, the basis described by $\{A_L^+ U_{N,L+1}\}$ does not span the whole space $L^2(R)$. A_L^+ is the operator that allows us to get the eigenfunctions of $\mathscr{H}_L$ from the corresponding eigenfunctions of H_{L+1}, but in this process we miss the real ground state for fixed L. We can solve this problem through the following considerations. The 'missing vector' $\tilde{U}_{L,L}(R)$, that completes the basis, is orthogonal to the set $\{A_L^+ U_{N,L+1}\}$, i.e.

$$(\tilde{U}_{L,L}, A_L^+ U_{N,L+1}) = 0 \qquad \text{for all} \quad N. \tag{2.32}$$

From (2.32), it is also true that

$$(A_L \tilde{U}_{L,L}, U_{N,L+1}) = 0 \tag{2.33}$$

and, since $\{U_{N,L+1}\}$ is a basis, we obtain the following condition to determine $\tilde{U}_{L,L}$:

$$A_L \tilde{U}_{L,L} = 0. \tag{2.34}$$

This is a first-order differential equation with solution given by

$$\tilde{U}_{L,L}(R) = CR^L \exp\left(-\frac{m\omega}{2\hbar}R^2 - \int_0^R \mathrm{d}R\,\phi_L(R)\right) \tag{2.35}$$

which corresponds to the ground state associated with each fixed L. The corresponding eigenvalue is given by $(2L+D)\hbar\omega/2$, since

$$\begin{aligned}\mathscr{H}_L \tilde{U}_{L,L}(R) &= [A_L^+ A_L + \tfrac{1}{2}\hbar\omega(2L+D)]\tilde{U}_{L,L}(R)\\ &= \tfrac{1}{2}\hbar\omega(2L+D)\,\tilde{U}_{L,L}(R).\end{aligned} \tag{2.36}$$

In this way, we succeeded in generalising the D-dimensional harmonic oscillator Hamiltonian, defined in terms of the a_L and a_L^+ operators, to a new class of Hamiltonians $\mathscr{H}_L$, defined in terms of the generalised A_L and A_L^+ operators. The spectrum of both Hamiltonians is the same when we include the 'missing vectors'.

3. The Coulomb potential in arbitrary dimensions and generalised factorisation

In a way very similar to that of the previous section where we discussed the D-dimensional harmonic oscillator, we can also define creation and annihilation operators to obtain the d-dimensional Coulomb Hamiltonian in a factorised form.

The present eigenvalue problem is

$$H_l R_{n,l}(r) = E_n R_{n,l}(r) \tag{3.1}$$

where (Nieto 1979, Kostelecky *et al* 1985)

$$H_l = \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{[l+\frac{1}{2}(d-1)][l+\frac{1}{2}(d-1)-1]}{r^2} - \frac{2me^2}{\hbar^2}\frac{1}{r}\right) \tag{3.2}$$

whose eigenvalues are given by

$$E_n = -\frac{me^4}{2\hbar^2}[n+\tfrac{1}{2}(d-3)]^{-2} \tag{3.3}$$

with $n = n_r + l + 1$, $n_r = 0, 1, \ldots,$ and $d \geqslant 3$.

The factorised form of this Hamiltonian can be written as

$$H_l = b_l^+ b_l - \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \tag{3.4}$$

with

$$b_l^+ = \frac{\hbar}{\sqrt{2m}}\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{me^2}{\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-1} - \frac{l+\frac{1}{2}(d-1)}{r}\right) \tag{3.5a}$$

and

$$b_l = \frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{me^2}{\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-1} - \frac{l+\frac{1}{2}(d-1)}{r}\right). \tag{3.5b}$$

In order to determine the eigenfunctions of the d-dimensional Coulomb Hamiltonian, we consider the following two possible forms of factorisation:

$$H_l = b_l^+ b_l - \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \tag{3.6}$$

and

$$H_{l+1} = b_l b_l^+ - \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}. \tag{3.7}$$

The corresponding eigenvalue equations are

$$H_l R_{m_r+l+1,l}(r) = E_{n_r+l+1,l} R_{n_r+l+1,l}(r) \tag{3.8}$$

and

$$H_{l+1} R_{n_r+l+2,l+1}(r) = E_{n_r+l+2,l+1} R_{n_r+l+2,l+1}(r). \tag{3.9}$$

From these equations, we obtain the following relations:

$$b_l b_l^+ b_l R_{n_r+l+1,l}(r) = \left(E_{n_r+l+1,l} + \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}\right) b_l R_{n_r+l+1,l}(r) \tag{3.10}$$

and

$$b_l b_l^+ R_{n_r'+l+2,l+1}(r) = \left(E_{n_r'+l+2,l+1} + \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \right) R_{n_r'+l+2,l+1}(r). \tag{3.11}$$

By comparing these two last equations we are led to the following results:

$$b_l R_{n_r+l+1,l}(r) = R_{n_r'+l+2,l+1}(r) \tag{3.12}$$

and

$$E_{n_r+l+1,l} = E_{n_r'+l+2,l+1}. \tag{3.13}$$

From the equality (3.13) between energies, we obtain $n_r' = n_r - 1$ which allows us to recognise b_l as a ladder operator, i.e.

$$b_l R_{n_r+l+1,l}(r) = R_{n_r+l+1,l+1}(r). \tag{3.14}$$

As we have done in the previous section, we can obtain ladder relations that involve the operator b_l^+, namely

$$b_l^+ R_{n_r+l+1,l+1}(r) = R_{n_r+l+1,l}(r). \tag{3.15}$$

Now we will generalise the operators b_l and b_l^+ in such a way that they will still produce the same spectrum of the usual d-dimensional Coulomb Hamiltonian.

Let us define the generalised operators

$$B_l = \frac{\hbar}{\sqrt{2m}}\left(\frac{\mathrm{d}}{\mathrm{d}r} + F_l(r)\right) \tag{3.16a}$$

$$B_l^+ = \frac{\hbar}{\sqrt{2m}}\left(-\frac{\mathrm{d}}{\mathrm{d}r} + F_l(r)\right) \tag{3.16b}$$

and demand that their product reproduces the spectrum of H_{l+1}, i.e.

$$H_{l+1} = B_l B_l^+ - \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}. \tag{3.17}$$

In this case, the differential equation that defines our $F_l(r)$ is given by

$$F_l' + F_l^2 = \frac{[l+\frac{1}{2}(d-1)][l+\frac{1}{2}(d-1)+1]}{r^2} - \frac{2me^2}{\hbar^2}\frac{1}{r} + \frac{m^2e^4}{\hbar^4}[l+\tfrac{1}{2}(d-1)]^{-2} \tag{3.18}$$

since

$$H_{l+1} = \frac{\hbar^2}{2m}\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + F_l^2(r) + F_l'(r)\right] - \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}. \tag{3.19}$$

In order to find the general solution of (3.18), we write $F_l(r)$ as its known particular solution plus an unknown function $\varphi_l(r)$ as

$$F_l(r) = \frac{me^2}{\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} - \frac{l+\frac{1}{2}(d-1)}{r} + \varphi_l(r). \tag{3.20}$$

We get

$$\varphi_l(r) = \frac{r^{2l+d-1}\exp\{-(2me^2/\hbar^2)[l+\frac{1}{2}(d-1)]^{-1}r\}}{\Gamma + \int_0^r \mathrm{d}r\, r^{2l+d-1}\exp\{-(2me^2/\hbar^2)[l+\frac{1}{2}(d-1)]^{-1}r\}}. \tag{3.21}$$

In this case, to avoid singularities, we require that

$$\Gamma>0 \qquad \text{or} \qquad \Gamma<-(2l+d-1)!\{(2me^2/\hbar^2)[l+\tfrac{1}{2}(d-1)]^{-1}\}^{-2l-d}. \tag{3.22}$$

As before, we notice that the commutator

$$[B_l, B_l^+]=\frac{\hbar^2}{m}\left(\frac{l+\frac{1}{2}(d-1)}{r^2}+\varphi_l'(r)\right) \tag{3.23}$$

and hence the inverted order of the product $B_lB_l^+$ will also define another Hamiltonian. First consider that we can write

$$H_{l+1}=B_l^+B_l-\frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}+\frac{\hbar^2}{m}\frac{l+\frac{1}{2}(d-1)}{r^2}+\frac{\hbar^2}{m}\varphi_l'(r). \tag{3.24}$$

Now from (3.2) we have

$$H_{l+1}-\frac{\hbar^2}{m}\frac{l+\frac{1}{2}(d-1)}{r^2}=H_l \tag{3.25}$$

which is useful for rewriting (3.24) in terms of H_l and comparing it with the generalised form obtained from (3.4). The result is written as

$$H_l-\frac{\hbar^2}{m}\varphi_l'(r)=B_l^+B_l-\frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}. \tag{3.26}$$

We see that the order of the operators B_l^+ and B_l on the RHS do not define the same H_l, and in consequence, we are led to define another Hamiltonian $\mathcal{H}_l$ as

$$\mathcal{H}_l=H_l-\frac{\hbar^2}{m}\varphi_l'(r). \tag{3.27}$$

This new operator exists due to the non-commutativity of B_l^+ and B_l, and corresponds to a modified potential v_l given by

$$v_l(r)=-\frac{e^2}{r}-\frac{\hbar^2}{m}\varphi_l'(r). \tag{3.28}$$

From the eigenvalue equation for $\mathcal{H}_l$ it is not difficult to see that $\{B_l^+R_{n,l+1}\}$ are the eigenfunctions if $\{R_{n,l+1}\}$ are the corresponding eigenfunctions of H_{l+1}, and that $\mathcal{H}_l$ and H_{l+1} have the same eigenvalue E_n. Here, as in the harmonic oscillator case, we also have 'missing states' $\tilde{R}_{l+1,l}$. They are similarly determined from the orthogonality condition, for all n,

$$(\tilde{R}_{l+1,l}, B_l^+R_{n,l+1})=0 \tag{3.29}$$

or

$$(B_l\tilde{R}_{l+1,l}, R_{n,l+1})=0. \tag{3.30}$$

We thus obtain the equation that defines the 'missing vector', namely

$$B_l\tilde{R}_{l+1,l}=0 \tag{3.31}$$

which admits the solution

$$\tilde{R}_{l+1,l}=Cr^l\exp\left(-\frac{me^2}{\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-1}r-\int_0^r\mathrm{d}r\,\varphi_l(r)\right). \tag{3.32}$$

This missing state has the eigenvalue

$$-\frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \tag{3.33}$$

and completes the basis for $\mathscr{H}_l$.

Therefore, we have obtained a new d-dimensional potential (3.28) whose soluble Hamiltonian $\mathscr{H}_l$ has the same spectrum as H_l.

4. Supersymmetric interpretation of the ladder operators

We have written the harmonic oscillator and Coulomb Hamiltonians in terms of creation and annihilation operators. In this section, we will interpret these operators as supersymmetric charges. With this identification, we intend to show that all new Hamiltonians obtained by the combination of those ladder operators, including their generalised versions, correspond to obtaining their supersymmetric partners.

Supersymmetric quantum mechanics refers to systems for which there exist charges Q and Q^* obeying the following anticommutation relations:

$$\{Q, Q^*\} = H_{SS} \qquad \{Q, Q\} = 0 \qquad \{Q^*, Q^*\} = 0. \tag{4.1}$$

These charges are usually written in the form

$$Q^* = (p + iw'(x))\sigma^- \tag{4.2}$$

and

$$Q = (p - iw'(x))\sigma^+ \tag{4.3}$$

where $[x, p] = i$ and $\{\sigma^-, \sigma^+\} = 1$. The supersymmetric Hamiltonian H_{SS} is written as

$$H_{SS} = p^2 + [w'(x)]^2 + w''(x)\sigma_3 \tag{4.4}$$

where σ_3 is the Pauli matrix.

The function $w(x)$ is known as the superpotential. We use the following matrix representation for σ^- and σ^+:

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

Equation (4.4) can be written as

$$H_{SS} = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} \tag{4.6}$$

with

$$D^{\mp} = \pm d/dx + w'(x). \tag{4.7}$$

The eigenstates of the supersymmetric Hamiltonian H_{SS} are written as a vector,

$$\psi = \begin{pmatrix} \psi_n^{(+)} \\ \psi_n^{(-)} \end{pmatrix} \tag{4.8}$$

and, from (4.6) and $H_{SS}\psi = E\psi$, we can obtain (Cooper *et al* 1983, 1985)

$$D^+\psi_n^{(+)} = \sqrt{E}\,\psi_n^{(-)} \tag{4.9}$$

and

$$D^-\psi_n^{(-)}=\sqrt{E}\,\psi_n^{(+)} \tag{4.10}$$

which show how supersymmetry implies that the solutions with the same energy are paired.

In the harmonic oscillator case, let us take

$$H_L-\tfrac{1}{2}\hbar\omega(2L+D)=H_L^- \tag{4.11}$$

which now defines the Hamiltonian H_L^- with eigenvalue zero, for all L. In the present context, the realisation for H_L^- is

$$H_L^-\equiv a_L^+a_L. \tag{4.12}$$

This Hamiltonian has the eigenvalue $2\hbar\omega_n$, $n\geqslant 0$. Its supersymmetric partner H_L^+ is given by

$$H_L^+\equiv a_La_L^+=H_{L+1}-\tfrac{1}{2}\hbar\omega(2L+D-2) \tag{4.13}$$

with eigenvalue $2\hbar\omega(n+1)$. Therefore, they have the same spectrum except for the ground state. We identify the operators a_L and a_L^+ with D^- and D^+, respectively, which realise the supersymmetric algebra. And so, as done by Kostelecky *et al* (1985), the superpotential w can be directly obtained from the $D^{\mp}$ operators. The ground state can be taken from $D^-\psi_0^{(-)}=0$ as

$$\psi_0^{(-)}\sim\exp(-w). \tag{4.14}$$

On the other hand, the generalised Hamiltonian H_{L+1} defined by the operators A_L and A_L^+, in (2.17a),

$$H_{L+1}-\tfrac{1}{2}\hbar\omega(2L+D-2)=A_LA_L^+ \tag{4.15}$$

will correspond to our generalised H_L^+, according to (2.16) and (2.17b). The latter, when compared with (4.4), with $f_L\propto w'$, already exhibits the superpotential, as noticed by Nieto (1984). Equation (2.18), introduced by the generalised factorisation method, shows the general form that the superpotential must have in order to produce the harmonic oscillator spectrum, except for the zero ground state in this case. The other Hamiltonian

$$\begin{aligned}H_L^-&\equiv\mathscr{H}_L-\tfrac{1}{2}\hbar\omega(2L+D)=A_L^+A_L\\&=\frac{\hbar^2}{2m}\left(-\frac{d^2}{dR^2}+f_L^2-f'_L\right)\end{aligned} \tag{4.16}$$

defines its generalised supersymmetric partner, since

$$-\mathrm{i}Q^*\begin{pmatrix}\psi^{(+)}\\0\end{pmatrix}=D^+\sigma^-\begin{pmatrix}U_{N,L+1}\\0\end{pmatrix}=\begin{pmatrix}0\\A_L^+U_{N,L+1}\end{pmatrix}=\begin{pmatrix}0\\\psi^{(-)}\end{pmatrix} \tag{4.17}$$

and

$$\mathrm{i}Q\begin{pmatrix}0\\\psi^{(-)}\end{pmatrix}=D^-\sigma^+\begin{pmatrix}0\\A_L^+U_{N,L+1}\end{pmatrix}=\begin{pmatrix}A_LA_L^+U_{N,L+1}\\0\end{pmatrix}=\begin{pmatrix}U_{N,L+1}\\0\end{pmatrix}=\begin{pmatrix}\psi^{(+)}\\0\end{pmatrix}. \tag{4.18}$$

The corresponding supersymmetric Hamiltonian for the harmonic oscillator is

$$H_{\mathrm{SS}}=\begin{pmatrix}H_{L+1}-\tfrac{1}{2}\hbar\omega(2L+D-2)&0\\0&\mathscr{H}_L-\tfrac{1}{2}\hbar\omega(2L+D)\end{pmatrix}. \tag{4.19}$$

Our conclusions for the harmonic oscillator can be directly extended to the Coulomb potential. In this case, we have

$$H_l^- \equiv b_l^+ b_l = H_l + \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \tag{4.20}$$

and

$$H_l^+ \equiv b_l b_l^+ = H_{l+1} + \frac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2}. \tag{4.21}$$

Again, we recognise b_l^+ and b_l as D^+ and D^- respectively. By considering their generalised version, we are led to the supersymmetric Hamiltonian

$$H_{SS} = \begin{pmatrix} H_{l+1} + \dfrac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} & 0 \\ 0 & \mathcal{H}_l + \dfrac{me^4}{2\hbar^2}[l+\tfrac{1}{2}(d-1)]^{-2} \end{pmatrix}. \tag{4.22}$$

We can obtain the superpotential from the generalised operators B_l and B_l^+ through (3.16)-(3.18).

In this way, we succeed in establishing the complete relation between the generalised annihilation and creation operators with all possible supersymmetric charges. These are expressed in terms of the superpotentials which can be obtained from (2.20), for a spectrum of the harmonic oscillator type, and from (3.20) for one of the Coulomb potential type.

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Note added. A similar discussion of the topics in § 2 has been presented by Zhu (1987).

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